Donaldson–Thomas invariants in mathematics and physics

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BPS states and Donald–Thomas theory

Irreducible unitary representations (multiples) of the $4d, N = 2$ super Poincaré algebra fall into three classes:

- generic representations with mass $>$ |central charge|,
- special representations with mass $=$ |central charge| (BPS bound), so-called BPS multiples,
- "non-physical" ones.

Given a 4d physical theory with $N = 2$ super Poincaré symmetry, one wants to analyse the particle content by studying the irreducible subrepresentations in the Hilbert space of the theory.

If the theory is obtained from a 10d theory by "compactifying" along some Calabi–Yau 3-category, the BPS multiples in the remaining 4d theory should be determined by the CY 3-category.

Examples:

- bounded derived category of (compactly supported) coherent sheaves on a CY 3-fold,
- Fukaya category of Lagrangian submanifolds (possibly with extra structure) in a CY 3-fold,
- category of matrix factorizations associated to a Landau–Ginzburg potential,
- dg-modules over the Ginzburg dg-algebra of a quiver with potential.

Roughly:

$\mathcal{H}^{BPS} \approx \mathrm{H}^*$(Moduli space of (semi)stable objects in the CY 3-cat.)

However:

- Moduli spaces are very singular and their cohomology has bad properties (e.g. no Poincaré duality),
- Moduli spaces might not exist for generic stability conditions,
- Moduli spaces might change dramatically under variation of parameters.

Donaldson–Thomas theory provides a possibly better replacement for the right hand side above.

The Ginzburg algebra of a quiver with potential

Given a quiver $Q = (Q_0, Q_1)$ with potential W, we form the dg-algebra $\Gamma(Q, W)$ generated by the following generators:

- e_i of degree 0 and t_i of degree -2 for every vertex $i \in Q_0$,
- a_α of degree 0 and b_α of degree -1 for every arrow $\alpha : i \to j$ in Q_1,

relations:

- $e_i e_j = \delta_{ij} e_j$ and $\sum_{i \in Q_0} e_i = 1$,
- $e_i t_j = t_j e_i = \delta_{ij} t_j$,
- $e_j a_\alpha = a_\alpha e_i = a_\alpha$ and $e_i b_\alpha = b_\alpha e_j = b_\alpha$ iff $\alpha : i \to j$,
- $de_i = da_\alpha = 0$,
- $db_\alpha = (\partial W / \partial \alpha)|_{\alpha' \mapsto a_{\alpha'}}$,
- $dt_i = \sum_{\alpha : i \to j} b_\alpha a_\alpha - \sum_{\alpha : j \to i} a_\alpha b_\alpha$.

composition is 0 unless paths can be composed

Modules over the Ginzburg algebra

A $\Gamma(Q, W)$-dg-module is a bigraded vector space

$$V^* = \oplus_{i \in Q_0} V_i^* \text{ with a differential } d : V^* \longrightarrow V^{*+1}$$

preserving the Q_0-grading, along with operators $\hat{e}_i, \hat{t}_i, \hat{a}_\alpha, \hat{b}_\alpha$ satisfying the relations above with $\hat{e}_i$ being the projection onto V_i^*. In particular, for $\alpha : i \to j$

$$\begin{aligned} \hat{a}_\alpha : V^* \twoheadrightarrow V_i^* &\longrightarrow V_j^* \hookrightarrow V^*, \\ \hat{b}_\alpha : V^* \twoheadrightarrow V_j^* &\longrightarrow V_i^{*-1} \hookrightarrow V^{*-1}, \end{aligned}$$

and $\hat{t}_i : V^* \twoheadrightarrow V_i^* \longrightarrow V_i^{*-2} \hookrightarrow V^{*-2}$ for $i \in Q_0$.

Theorem (Ginzburg)

All $\Gamma(Q, W)$-dg-modules with finite dimensional cohomology form a Calabi–Yau 3-category.

Modules over the Jacobi algebra

For simplicity, we assume that V^* is concentrated in degree 0, i.e. we have a vector space $V = \oplus_{i \in Q_0} V_i$ and operators $\hat{a}_\alpha : V_i \longrightarrow V_j$ for $\alpha : i \to j$ such that

$$(\partial W / \partial \alpha)|_{\alpha' \mapsto \hat{a}_{\alpha'}} = 0$$

for all $\alpha \in Q_1$. $\hat{e}_i : V \twoheadrightarrow V_i$ are the projection operators as before.

Such a module can be seen as a module over the Jacobi algebra $J(Q, W) := \mathrm{H}^0\,\Gamma(Q, W)$, i.e. the path algebra of the quiver Q generated by e_i and a_α modulo the relations $(\partial W / \partial \alpha)|_{\alpha' \mapsto a_{\alpha'}} = 0$ for all $\alpha \in Q_1$.

Proposition

Finite dimensional $J(Q, W)$-modules form the heart of a bounded t-structure on the CY 3-category mentioned before.

We want to construct a moduli "space" of all $J(Q,W)$-modules of fixed dimension vector $d = (d_i = \dim V_i)_{i \in Q_0}$. Since $V_i \cong \mathbb{C}^{d_i}$, we need to choose operators $\hat{a}_\alpha : \mathbb{C}^{d_i} \longrightarrow \mathbb{C}^{d_j}$ for all $\alpha : i \to j$.

Space of choices:

$$R_d = \prod_{\alpha:i\to j} \mathrm{Mat}(d_j, d_i) = \mathbb{C}^{\sum_{\alpha:i\to j} d_i d_j}$$

The group $G_d = \prod_{i \in Q_0} \mathrm{GL}_{d_i}$ acts on R_d by simultaneous conjugation. (change of isomorphism $V_i \cong \mathbb{C}^{d_i}$)

There is a mathematical object, called the quotient stack,

$$\mathfrak{U}_d = R_d / G_d$$

parametrising modules over the path algebra and their isomorphisms (including automorphisms).

Critical loci and vanishing cycles

The potential W defines a function $W_d : R_d \longrightarrow \mathbb{C}$ by $W_d((\hat{a}_\alpha)_{\alpha \in Q_1}) := \mathrm{Tr}(W|_{\alpha \mapsto \hat{a}_\alpha})$. Observation: $\mathrm{crit}\, W_d \subset R_d$ is the subspace of matrices $\hat{a}_\alpha$ satisfying $(\partial W / \partial \alpha)|_{\alpha' \mapsto \hat{a}_{\alpha'}} = 0$, i.e.

$$\mathfrak{M}_d = \mathrm{crit}\, W_d / G_d$$

parametrises $J(Q, W)$-modules and their isomorphisms (including automorphisms).

Moreover: There is a "perverse sheaf" ϕ_{W_d} supported on $\mathrm{crit}\, W_d$, the vanishing cycle sheaf.

Example: For $W = 0$, ϕ_{W_d} is the constant sheaf $\underline{\mathbb{C}}_{R_d}$ on R_d with fiber $\mathbb{C}$ (shifted in a certain degree).

Stability

Fix a “stability condition” $z = (z_i)_{i \in Q_0}$ with $z_i \in \mathbb{C}$ such that $\operatorname{Im} z_i > 0$ for all $i \in Q_0$.

Semistable modules V are defined by

$$\arg \sum_{i \in Q_0} z_i \dim V'_i \leq \arg \sum_{i \in Q_0} z_i \dim V_i$$

for every proper module $V' \subset V$. Let $R_d^{ss} \subset R_d$ be the open subspace of semistable modules of the path algebra.

Example: For $z_i = \theta_i + \sqrt{-1}$ with $\theta_i \in \mathbb{Z}$, this is King's stability and there is a GIT quotient $R_d^{ss} /\!\!/ G_d$, the moduli space of semistable quiver representations.

Defining Donaldson–Thomas invariants

Fix a stability condition $z = (z_i)_{i \in Q_0}$ and an angle $\gamma \in (0, \pi)$.

Consider the bigraded vector space:

$$A(Q, W, z)^*_\gamma := \bigoplus_{\substack{d \in \mathbb{N}^{Q_0} \\ \arg \sum_i z_i d_i = \gamma}} \mathrm{H}^*_{G_d}(R^{ss}_d, \phi_{W_d})$$

Example: For $W = 0$ this is

$$A(Q, 0, z)^*_\gamma = \bigoplus_{\substack{d \in \mathbb{N}^{Q_0} \\ \arg \sum_i z_i d_i = \gamma}} \mathrm{H}^{*+\cdots}_{G_d}(R^{ss}_d, \mathbb{C})$$

There is a bigraded vector space

$$DT(Q,W,z)^*_\gamma = \bigoplus_{\substack{d\in\mathbb{N}^{Q_0}\setminus\{0\} \\ \arg\sum_i z_i d_i=\gamma}} DT(Q,W,z)^*_d$$

such that

$$A(Q,W,z)^*_\gamma \cong \mathrm{Sym}\left(DT(Q,W,z)^*_\gamma \otimes \mathrm{H}^{*+1}_{\mathbb{C}^*}(pt)\right) \text{ holds.}$$

Given a dimension vector $d=(d_i)_{i\in Q_0}$, take $\gamma := \arg\sum_i z_i d_i$ and read off $DT(Q,W,z)^*_d$ from $DT(Q,W,z)^*_\gamma$. This is called the Donaldson–Thomas invariant for dimension vector d of (Q,W) with respect to the stability condition z.

Conjecture (Integrality Conjecture)

For generic stability conditions $z=(z_i)_{i\in Q_0}$, the graded vector spaces $DT(Q,W,z)^*_d$ are finite dimensional.

Example

Fix $W = 0$ and $z_i = \theta_i + \sqrt{-1}$ with $\theta_i \in \mathbb{Z}$ (King's stability) such that $\sum_{\alpha:i\to j} d_i d'_j = \sum_{\alpha:i\to j} d'_i d_j$ if $\arg \sum_i z_i d_i = \arg \sum_i z_i d'_i$.
Form the (mostly singular) GIT quotient

$$M_d^{ss} := R_d^{ss} /\!\!/ G_d \quad \text{and let} \quad M_d^{st} \subset M_d^{ss}$$

be the subspace of stable modules (might be empty).

Theorem (Reineke, -)

$$DT(Q,0,z)_d^* = \begin{cases} \mathrm{IH}^*(M_d^{ss}, \mathbb{C}) & \textit{if } M_d^{st} \neq \emptyset \\ 0 & \textit{else.} \end{cases}$$

Here $\mathrm{IH}^*(-, \mathbb{C})$ is the intersection cohomology of a (singular) space. Intersection cohomology improves the "normal" cohomology for singular spaces, e.g. Poincaré duality holds.

This looks like an improved version for

$\mathcal{H}^{BPS} \approx \mathrm{H}^*$(Moduli space of (semi)stable objects in the CY 3-cat.)

However, a similar result cannot hold for $W \neq 0$ as the example of the 3 loop quiver $Q_{(3)}$ with potential $W = xyz - xzy$ shows.

Theorem (Behrend, Bryan, Szendroi)

$$DT(Q_{(3)}, W, z)^*_d = \mathrm{H}^{*+3}(\mathbb{C}^3, \mathbb{C}) = \mathbb{C}[3] \textit{ for all } d \geq 1.$$

Note that $M^{ss}(Q_{(3)}, W)_d = \mathrm{Sym}^d\,\mathbb{C} = (\mathbb{C}^3)^d/S_d$ but $M^{st}(Q_{(3)}, W)_d = \emptyset$ for all $d > 1$.

Question: What is the correct geometric interpretation of $DT(Q, W, z)^*_d$ for all quiver Q with potential W?